On the Set of All Stabilizing First-Order Controllers

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Abstract

A computational method is given for determining the set of all stabilizing proper first-order controllers for finite dimensional, linear, time invariant, scalar plants. The method is based on a generalized Hermite-Biehler theorem.

Keywords: Hermite-Biehler theorem, stabilization, linear systems.

1 Introduction

In [1], a computational characterization of all stabilizing proportional-integral (PI) and proportional-integral-derivative (PID) controllers was derived. In [2], an alternative fast method for determining all stabilizing PID controllers was derived. The limiting values of controller parameters that guarantee stability are determined in [1] using an extension of the Hermite-Biehler theorem [3], and using the Nyquist plot in [2]. In this paper, we give an extension of the method of [1] to solve the problem of determining all first-order controllers that stabilizes a given plant. The paper is organized as follows. In section 2, the method for calculating stabilizing gains is revisited. In section 3, we give an algorithm for determining stabilizing first-order controllers. Finally, section 4 contains some concluding remarks.

2 Proportional Controllers

Let $\mathbf{C}$ denote the set of complex numbers and let $\mathbf{C}_-$, $\mathbf{C}_0$, $\mathbf{C}_+$ denote the points in the open left half, $j\omega$-axis, and the open right half of the complex plane, respectively. Then, the set $\mathcal{H}$ of Hurwitz stable polynomials are $\mathcal{H} = \{\psi(s) \in \mathbf{R}[s] : \psi(s) = 0 \Rightarrow s \in \mathbf{C}_-\}$. The derivative of ψ is denoted by ψ' and the *signature $\sigma(\psi)$ of a polynomial $\psi \in \mathbf{R}[s]$* is the difference between the number of its $\mathbf{C}_-$ roots and $\mathbf{C}_+$ roots. Given $\psi \in \mathbf{R}[s]$, *the even-odd components (a, b) of $\psi(s)$* are the unique polynomials $a, b \in \mathbf{R}[u]$ such that $\psi(s) = a(s^2) + sb(s^2)$. Finally, the greatest common divisor of a and b is denoted by $gcd\{a, b\}$ and $\lfloor m \rfloor$ denotes the greatest integer less than or equal to $m \in \mathbf{R}$. We state the following result for later reference. The proof of this result follows by a generalization of the Hermite-Biehler theorem [3].

Lemma 1. *A nonzero polynomial $\psi \in \mathbf{R}[s]$ has r real negative roots without counting the multiplicities if and only if the signature of the polynomial $\psi(s^2) + s\psi'(s^2)$ is $2r$. All roots of ψ are real, negative, and distinct if and only if $\psi(s^2) + s\psi'(s^2) \in \mathcal{H}$.*

We now describe a slight extension of the constant stabilizing gain algorithm of [3]. Given a plant $g(s) = \frac{p(s)}{q(s)}$, where $p, q \in \mathbf{R}[s]$ are nonzero with $m = \deg p$ less than or equal to $n = \deg q$, the set $A_r(p, q) := \{\alpha \in \mathbf{R} : \sigma[\phi(s, \alpha)] = \sigma[q(s) + \alpha p(s)] = r,\ deg\ \phi = deg\ q\}$, is the set of all real α such that $\phi(s, \alpha)$ has signature equal to r. Let (h, g) and (f, e) be the even-odd components of q and p, respectively. Let $d := gcd\{f, e\}$ so that $f = d\bar{f}$, $e = d\bar{e}$, or coprime polynomials $\bar{f}$, $\bar{e} \in \mathbf{R}[u]$. Then, the polynomial $\bar{p}(s) := \bar{f}(s^2) + s\bar{e}(s^2) = p(s)/d(s^2)$ is free of $\mathbf{C}_0$ roots except possibly a simple root at $s = 0$. Let (H, G) be the even-odd components of $q(s)\bar{p}(-s)$. Also let $F(s^2) := p(s)\bar{p}(-s)$. By a simple computation, it follows that $H(u) = h(u)\bar{f}(u) - ug(u)\bar{e}(u)$, $G(u) = g(u)\bar{f}(u) - h(u)\bar{e}(u)$, and $F(u) = f(u)\bar{f}(u) - ue(u)\bar{e}(u)$. If $G \not\equiv 0$ and if they exist, let the *real negative zeros with odd multiplicities of $G(u)$* be $\{v_1, ..., v_k\}$ with the ordering $v_1 > v_2 > \cdots > v_k$, with $v_0 := 0$ and $v_{k+1} := -\infty$ for notational convenience. The following algorithm determines whether $A_r(p, q)$ is empty or not and outputs its elements when it is not empty:
Algorithm 1.

1. *Calculate*

$$\alpha_i = \begin{cases} -\frac{H}{F}(v_i), i = 0, \ldots, k & \textit{for odd } r - m \\ -\frac{H}{F}(v_i), i = 0, \ldots, k+1 & \textit{for even } r - m, \end{cases}$$

   and sort them in ascending order $\bar{\alpha}_0 < \bar{\alpha}_1 < \ldots < \bar{\alpha}_{k+2} < \bar{\alpha}_{k+3}$ where $\bar{\alpha}_0 = -\infty$ and $\bar{\alpha}_{k+3} = \infty$.

2. *Identify all the sequences of signums*

$$\mathcal{I} = \begin{cases} \{i_0, i_1, \ldots, i_k\} & \textit{for odd } r - m \\ \{i_0, i_1, \ldots, i_{k+1}\} & \textit{for even } r - m, \end{cases}$$

   where $i_0 \in \{-1, 0, 1\}$ and $i_j \in \{-1, 1\}$ for $j = 1, \ldots, k+1$, that correspond to the intervals $(\bar{\alpha}_i, \bar{\alpha}_{i+1})$ for $i = 0, \ldots, k+2$.

3. *For each signum sequence $\mathcal{I}_i$ from step 2, if*

$$r - \sigma(p) = \begin{cases} i_0 - 2i_1 + \cdots + 2(-1)^k i_k & r - m \textit{ odd} \\ i_0 - \cdots + (-1)^{k+1} i_{k+1} & r - m \textit{ even.} \end{cases}$$

   holds, then $(\bar{\alpha}_i, \bar{\alpha}_{i+1}) \in A_r(p, q)$

Remark 1. By step 3 of Algorithm 1, a necessary condition for the existence of an $\alpha \in A_r(p, q)$ is that the odd part of $[q(s) + \alpha p(s)]\bar{p}(-s)$ has at least $\bar{r} = \max\{0, \lfloor \frac{|r-\sigma(p)|-1}{2} \rfloor\}$ real negative roots with odd multiplicities. When solving a constant stabilization problem this lower bound is $\bar{r} = \max\{0, \lfloor \frac{|n-\sigma(p)|-1}{2} \rfloor\}$.

3 First-Order Controllers

A first-order controller $c(s) = \frac{\alpha_2 s+\alpha_3}{s+\alpha_1}$ applied to $g_0(s) = \frac{p_0(s)}{q_0(s)}$, with $m = \deg p_0$ less than or equal to $n = \deg q_0$, gives the closed loop characteristic polynomial

$$\begin{aligned}\phi_1(s,\alpha_1,\alpha_2,\alpha_3) &= (s+\alpha_1)q_0(s) + (\alpha_2 s + \alpha_3)p_0(s) \\ &= q_1(s) + \alpha_3 p_1(s) \qquad (1)\end{aligned}$$

Multiplying $\phi_1(s,\alpha_1,\alpha_2,\alpha_3)$ by $\bar{p}_1(-s)$ we obtain

$$\begin{aligned}\psi_1(s,\alpha_1,\alpha_2,\alpha_3) &= \phi_1(s,\alpha_1,\alpha_2,\alpha_3)\bar{p}_1(-s) \\ &= s^2G(s^2) + \alpha_1 H(s^2) + \alpha_3 F(s^2) \\ &\quad + s[H(s^2) + \alpha_1 G(s^2) + \alpha_2 F(s^2)]. \qquad (2)\end{aligned}$$

Note that α_1, α_2 appear in the odd part and α_1, α_3 appear in the even part. It is no longer possible to exploit the results given in [1] and proceed.

The reasoning behind the below algorithm can be explained as follows. Suppose that $\phi_1(s)$ is Hurwitz stable for some α_1, α_2, $\alpha_3 \in \mathbf{R}$. By Remark 1, it follows that the odd part of $\psi_1(s)$ has at least $r_1 = \lfloor \frac{n-\sigma(p_1)}{2} \rfloor$ real negative roots with odd multiplicities. By Lemma 1, $\sigma[\phi_2(s)] = 2r_1$, where

$$\begin{aligned}\phi_2(s) &= H(s^2) + \alpha_1 G(s^2) + \alpha_2 F(s^2) \\ &\quad + s[H'(s^2) + \alpha_1 G'(s^2) + \alpha_2 F'(s^2)] \\ &= q_2(s) + \alpha_2 p_2(s)\end{aligned}$$

In order to find the suitable ranges of α_1 and α_2, we modify $\phi_2(s)$ as follows.. Let $B := gcd\{F, F'\}$ so that $F = B\bar{F}$, $F' = B\bar{F}'$ for coprime polynomials $\bar{F}, \bar{F}' \in \mathbf{R}[u]$. Let $p_2(s) := \bar{F}(s^2) + s\bar{F}'(s^2)$. Then, by a straightforward computation, $\psi_2(s) = \phi_2(s)\bar{p}_2(-s) = I(s^2) + \alpha_1 J(s^2) + \alpha_2 K(s^2) + s[L(s^2) + \alpha_1 M(s^2)]$, where $I(u)$, $J(u), K(u)$, $L(u)$, and $M(u)$ can be easily computed. Once more by Remark 1, the odd part of $\psi_2(s)$ has at least $r_2 = \lfloor \frac{|2r_1-\sigma(p_2)|-1}{2} \rfloor$ real negative roots with odd multiplicities. Values of $\alpha_1 \in \mathbf{R}$ achieving r_2 real negative roots with odd multiplicities can be determined using Lemma 1 and Algorithm 1. In Algorithm 2 below, we follow a nested procedure where one α_1, α_2 is determined and then the corresponding values of α_3 are obtained.

Algorithm 2.

1. *Using Lemma 1 and Algorithm 1, partition the real axis into intervals such that in every interval $L(u) + \alpha_1 M(u)$ has a constant number of real negative roots with odd multiplicities.*
2. *Fix $r_1 = \lfloor \frac{n-\sigma(p_1)}{2} \rfloor$.*
   (a) *Determine from step 1 if a range of α_1 for which $r_2 = \lfloor \frac{|2r_1-\sigma(p_2)|-1}{2} \rfloor$ exists.*
      i. *Fix an α_1 in the range of step 2.a.*
      ii. *Apply Algorithm 1 to $q_2(s)$ replacing $q(s)$, $p_2(s)$ replacing $p(s)$, and $\sigma(\phi_2) = 2r_1$ replacing r. (This calculates admissible values of α_2 such that $H(u)+\alpha_1 G(u)+\alpha_2 F(u)$ has r_1 real negative roots with odd multiplicities.)*
         A. *Fix an α_2 from the range determined in 2.a.ii.*
         B. *Apply Algorithm 1 to $q_1(s)$ and $p_1(s) = p(s)$. (This calculates all admissible values of α_3 such that ϕ_1 of (1) is in $\mathcal{H}$.)*
         C. *Increment α_2 and go to step 2.a.ii.A.*
      iii. *Increment α_1 and go to step 2.a.i.*
   (b) *Increment r_1 by one, if $r_1 \leq deg(H(u) + \alpha_1 G(u) + \alpha_2 F(u))$ go to step 2.a.*

Example 1. Consider determining proper first-order controllers to stabilize the plant $g_0(s) = \frac{p_0(s)}{q_0(s)}$ given in [1], where $q_0(s) = s^5 + 3s^4 + 29s^3 + 15s^2 - 3s + 60$ and $p_0(s) = s^3 - 6s^2 + 2s + 1$. Algorithm 2 outputs the set of stabilizing $(\alpha_1, \alpha_2, \alpha_3)$ values as shown in Figure 1.

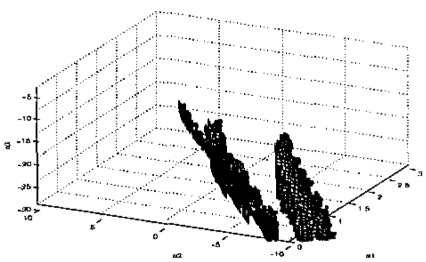

Figure 1: Stabilizing set of $(\alpha_1, \alpha_2, \alpha_3)$ values.

4 Conclusions

In this paper a solution is given to the problem of determining all first-order controllers that stabilize a given plant. The method consists of an application of constant stabilizing gain characterization on two auxiliary plants. An extension of this method to any fixed order controller is reported in [5] and to interval plants is reported in [4].

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